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Nonlinear Kinematic Hardening under Non-Proportional Loading

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NONLINEAR KINEMATIC HARDENING UNDER NON-PROPORTIONAL LOADING

N.S. Ottosen

Abstract. Within the framework of conventional plasticity theory, it is first determined under which conditions Melan-Prager's and Ziegler's kinematic hardening rules result in identical material behaviour. Next, assuming initial isotropy and adopting the von Mises yield criterion, a nonlinear kinematic hardening function is proposed for prediction of metal behaviour. The model assumes that hardening at a specific stress point depends on the direction of the new incremental loading. Hereby a realistic response is obtained for general reversed loading, and a smooth behaviour is assured, even when loading deviates more and more from proportional loading and ultimately results in reversed loading. The predictions of the proposed model for non-proportional loading under plane stress conditions are compared with those of the classical linear kinematic model, the isotropic model and with published experimental data. Finally, the limitations of the proposed model are discussed.

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1. INTRODUCTION

The accurate calculation of non-proportional inelastic behaviour, including the cycling of metals under multi-axial stress states, is of importance for many structures, and notably for structures with aerospace and nuclear applications. For this purpose the adoption of kinematic hardening, within classical plasticity theory, seems to offer promising possibilities. In this paper it is shown that the only cases where Melan-Prager's and Ziegler's kinematic hardening rules result in the same material behaviour are the cases where the initial yield surface is a sphere in the stress space and where it is a cylinder with a circular cross section. For the latter yield surface, the hardening function must be of a certain type if this coincidence is to exist. Assuming initial isotropy and adopting the von Mises criterion, a nonlinear kinematic hardening function of this type, calibrated by any uniaxial stress-strain curve, is then proposed for the prediction of metal behaviour. The model implies a realistic response for general reversed loading, and a smooth behaviour is obtained when loading deviates more and more from proportional loading and ultimately results in reversed loading. The predictions of the proposed model for non-proportional loading under plane stress conditions are compared with those of the classical linear kinematic model, the isotropic model and with published experimental data obtained for stainless steel. Finally, the limitations of the model used in the present paper are discussed.

2. KINEMATIC HARDENING

It is commonly known that for loadings that are far from proportional, isotropic hardening is insufficient, and kinematic hardening, where the loading surfaces translate as rigid surfaces maintaining their orientation in the stress space, provides an approximation to reality that seems more promising. In particular, kinematic hardening provides means for consideration of the Bauschinger effect observed in the behaviour of most metals. If the function $f(\sigma_{ij})$, symmetric in the components σ_{ij}, is used to describe the initial yield surface of the material, i.e.

$$f(\sigma_{ij}) = \kappa, \tag{2.1}$$

where σ_{ij} is the stress tensor referred to a fixed rectangular cartesian coordinate system, and κ is a constant, then under the assumption of kinematic hardening, the loading surfaces are given by

$$f(\sigma_{ij}-\alpha_{ij}) = \kappa. \tag{2.2}$$

Here f is the same function as in (2.1), and the symmetric tensor α_{ij} describes the total translation of the centre of the loading surface in the stress space. Naturally, the way in which α_{ij} and the plastic history are related is the crucial point in any kinematic hardening theory. Figure 1 illustrates the translation of the loading surface from position 1 to position 2 due to hardening. O is the origin of the stress

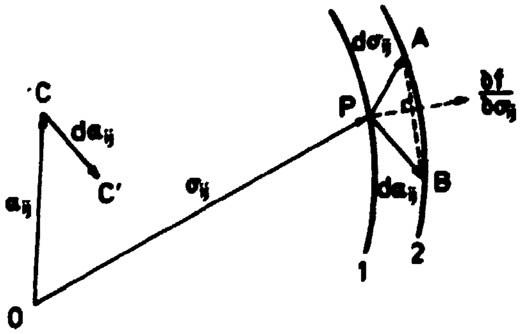

Fig. 1. Translation of loading surface.

space, and C is the centre of the loading surface 1, which shifts to C' during hardening. P denotes the actual stress point located on loading surface 1, whereas point A is located on loading surface 2 with the centre C'. Point B is also located on surface 2 as a result of (2.2). If we accept the normality condition arising, e.g. from Drucker's postulate for stable material behaviour, DRUCKER (1951), then during loading

$$d\varepsilon_{ij}^{p} = d\lambda\frac{\partial f}{\partial \sigma_{ij}} , \tag{2.3}$$

where $d\varepsilon_{ij}^{p}$ denotes the differential of the plastic strain tensor, and $d\lambda$ is a positive scalar function. Projecting $d\sigma_{ij}$ on the normal at point P, given by $\partial f/\partial\sigma_{ij}$, we obtain the scalar product $d\sigma_{ij}\ \partial f/\partial\sigma_{ij}$, which can always be set equal to the scalar product of the two proportional vectors $d\varepsilon_{ij}^{p}$ and $\partial f/\partial\sigma_{ij}$ multiplied by a suitable positive factor. This leads to

$$(d\sigma_{ij}-c\,d\varepsilon_{ij}^{p})\frac{\partial f}{\partial \sigma_{ij}} = 0, \tag{2.4}$$

where c is a hardening function depending in general on the loading history and the present incremental loading. Rearranging (2.4), we find

$$d\varepsilon_{ij}^{p}\frac{\partial f}{\partial \sigma_{ij}} = \frac{1}{c}\,\frac{\partial f}{\partial \sigma_{kl}}\,d\sigma_{kl}. \tag{2.5}$$

Elimination of $d\varepsilon_{ij}^{p}$ by means of (2.3) implies

$$d\lambda = \frac{1}{c}\,\frac{(\partial f/\partial\sigma_{kl})d\sigma_{kl}}{(\partial f/\partial\sigma_{ij})(\partial f/\partial\sigma_{ij})} , \tag{2.6}$$

i.e. $d\lambda$ is determined by the hardening function c once the loading function is known. It now remains to complete the equations required by determining the tensor $d\alpha_{ij}$. Using (2.2), the consistency equation states that

$$(d\sigma_{ij}-d\alpha_{ij})\frac{\partial f}{\partial \sigma_{ij}} = 0. \tag{2.7}$$

Thus, (2.7) determines the projection of $d\alpha_{ij}$ on the normal at point P, and $d\alpha_{ij}$ is then fully known once the direction of $d\alpha_{ij}$ is chosen. MELAN (1938) and PRAGER (1955, 1956) assumed that the instantaneous translation of the loading surface was orthogonal to the surface at the stress point, which means that $d\alpha_{ij}$ is proportional to $d\varepsilon_{ij}^{p}$. Use of (2.4) and (2.7) then gives Melan-Prager's hardening rule:

$$d\alpha_{ij} = c\,d\varepsilon_{ij}^{p} \,, \tag{2.8}$$

where c was considered a constant in Melan-Prager's concept. ZIEGLER (1959) proposed another hardening rule, where $d\alpha_{ij}$ is assumed to be in the direction of the vector CP connecting the centre C of the loading surface with the actual stress point P, Fig. 1. Ziegler's hardening rule is therefore given by

$$d\alpha_{ij} = (\sigma_{ij}-\alpha_{ij})d\mu, \tag{2.9}$$

where the scalar function $d\mu$ is positive. Elimination of $d\alpha_{ij}$ in (2.7) by means of (2.9) and use of (2.4) leads to

$$d\mu = \frac{c\,d\varepsilon_{ij}^{p}(\partial f/\partial\sigma_{ij})}{(\sigma_{kl}-\alpha_{kl})(\partial f/\partial\sigma_{kl})} \tag{2.10}$$

As shown by PERRONE and HODGE (1958, 1959), Melan-Prager's hardening rule, (2.8), must always be applied in the full 9-dimensional stress space, as (2.8) is not invariant with respect to reductions in dimensions. Even if some components of σ_{ij} are equal to zero, the corresponding components of $d\varepsilon_{ij}^{p}$ and $d\alpha_{ij}$ are in general non-zero. This is not the case for Ziegler's hardening rule, (2.9), which therefore has the advantage of being invariant and is thus more attractive from a mathematical point of view. Detailed discussions of (2.8) and (2.9) are given by SHIELD and ZIEGLER (1958), ZIEGLER (1959), CLAVOUT and ZIEGLER (1959) and NAGHDI (1960).

Let us now investigate the cases where Melan-Prager's and Ziegler's hardening rules result in the same material behaviour.

For this purpose we define the reduced stress tensor σ'_{ij} by

$$\sigma'_{ij} = \sigma_{ij} - \alpha_{ij}.$$

If we assume that the loading surface (2.2) is not a cylinder in the stress space, it involves all the components of σ'_{ij}, and to ensure that the two hardening rules result in identical material behaviour, we must require the differentials $d\alpha_{ij}$ to be identical. Use of (2.3), (2.8) and (2.9) yields

$$c d\lambda \frac{\partial f}{\partial \sigma_{ij}} = \sigma'_{ij}\, d\mu,$$

and, as $\partial f/\partial \sigma_{ij} = \partial f/\partial \sigma'_{ij}$, we obtain

$$f(\sigma_{ij}-\alpha_{ij}) = A\sigma'_{ij}\, \sigma'_{ij} = \kappa, \tag{2.11}$$

where A is a positive scalar. The above equation shows that the initial yield surface is given by a sphere in the stress space.

If the direction of a line is given by l_{ij}, where l_{ij} is defined such that $l_{ij}l_{ij} = 1$, then the component of any tensor d_{ij} in a plane perpendicular to the direction of l_{ij} is given by

$$d^{n}_{ij} = d_{ij} - d_{kl}\, l_{kl}\, l_{ij}, \tag{2.12}$$

where index n indicates that the component is normal to the direction of l_{ij}. Assume now that the loading surface is a cylinder in the stress space, and that the direction of the cylinder axis is given by l_{ij}. In this case the loading function only involves the components σ^{n}_{ij} and α^{n}_{ij}, or more precisely only the component $\sigma'^{n}_{ij} = \sigma^{n}_{ij} - \alpha^{n}_{ij}$. If the two hardening rules are to result in identical material behaviour for a given stress history, we must require that the components $d\alpha^{n}_{ij}$ of both hardening rules coincide. Use of (2.3), (2.8), (2.9) and (2.12) then leads to

$$c d\lambda \left(\frac{\partial f}{\partial \sigma_{ij}} - \frac{\partial f}{\partial \sigma_{kl}}\, l_{kl}\, l_{ij}\right) = \sigma'^{n}_{ij}\, d\mu,$$

and as $l_{kl}\, \partial f/\partial \sigma_{kl}$ is equal to zero, we obtain

$$\frac{\partial f}{\partial \sigma_{ij}} = 2\, B\, \sigma'^{n}_{ij},$$

where B is a positive scalar. As $\partial f/\partial\sigma_{ij} = \partial f/\sigma'^{n}_{ij}$, integration yields

$$f(\sigma_{ij}-\alpha_{ij}) = B\,\sigma'^{n}_{ij}\,\sigma'^{n}_{ij} = \kappa. \qquad (2.13)$$

I.e. the initial cylindrical yield surface intersects a plane perpendicular to the cylinder axis in a circular curve. The two hardening rules (2.8) and (2.9) result in identical material behaviour if the loading function is described by (2.13), but it should be noted that the translation tensor α_{ij} may have a component in the direction of l_{ij}, if Ziegler's hardening rule is applied. Therefore, to ensure coincidence of the material behaviour predicted by use of (2.8) or (2.9), we must also require that the hardening function c is not a function of the component of the translation tensor α_{ij} in the direction of the cylinder axis.

In summary, we have shown that the only cases where Melan-Prager's and Ziegler's hardening rules result in identical material behaviour are those where the loading surface is a sphere in the stress space, (2.11), or where it is a cylindrical surface with a circular cross section, (2.13). In the latter case, the hardening function c must not depend on the component of the translation tensor α_{ij} in the direction of the cylinder axis. If initial isotropy is assumed, (2.13) corresponds to the use of the von Mises criterion. For the spe ial cases of plane stress and plane strain when applying the von Mises criterion and considering c as a constant, the above coincidence has earlier been shown by CLAVOUT and ZIEGLER (1959).

Assuming initial isotropy, we will in the following restrict ourselves to use of the von Mises criterion, as it often represents the initial yielding of metals with sufficient accuracy, and as it is mathematically attractive both because of its lack of singularities and because of the resulting identical material behaviour whether Melan-Prager's or Ziegler's hardening rule is applied. The choice between the two hardening rules, can therefore be based on mathematical convenience, and here Ziegler's hardening rule seems to have some advantages, as pointed out earlier. Define the deviatoric stress tensor s_{ij} by

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\,\delta_{ij}\,\sigma_{kk}$$

- cf. (2.12) with $l_{ij} = \delta_{ij}/\sqrt{3}$ - and denote the yield stress for uniaxial tensile loading by σ_o, then (2.2) becomes

$$f(\sigma_{ij}-\alpha_{ij}) = (\tfrac{3}{2}\, s'_{ij}\, s'_{ij})^{\frac{1}{2}} = \sigma_o, \tag{2.14}$$

where the deviatoric translation tensor α^D_{ij} and the reduced deviatoric stress tensor s'_{ij} are defined by

$$\alpha^D_{ij} = \alpha_{ij} - \frac{1}{3}\,\delta_{ij}\,\alpha_{kk}$$

$$s'_{ij} = \sigma'_{ij} - \frac{1}{3}\,\delta_{ij}\,\sigma'_{kk} = s_{ij} - \alpha^D_{ij}\ .$$

Use of (2.3) and (2.14) in (2.10) leads to

$$d\mu = \frac{3}{2\sigma_o}\, c\, d\lambda;$$

while use of (2.14) in (2.6) yields

$$d\lambda = \frac{1}{c\sigma_o}\, s'_{ij}\, d\sigma_{ij}.$$

The last equation implies that

$$d\varepsilon^P_{ij} = \frac{3}{2c\sigma_o^2}\, s'_{ij}\, s'_{kl}\, d\sigma_{kl} \tag{2.15}$$

and

$$d\mu = \frac{3}{2\sigma_o^2}\, s'_{ij}\, d\sigma_{ij}.$$

Let the indices MP and Z refer to Melan-Prager's and Ziegler's hardening rule, respectively. From the last two equations we can then easily establish the expected relations, namely that $d\alpha^D_{ij,Z} = d\alpha^D_{ij,MP} = d\alpha_{ij,MP}$, which means that $s'_{ij,MP} = s'_{ij,Z}$ and therefore also that $d\varepsilon^P_{ij,MP} = d\varepsilon^P_{ij,Z}$ holds for a given stress history, provided that the hardening function c does not depend on the hydrostatic component of α_{ij}.

3. ELEMENTS OF NONLINEAR HARDENING

To obtain an interpretation of the hardening function c, multiply (2.5) by $d\lambda$ and use (2.3), and then

$$d\varepsilon^p_{ij} \; d\varepsilon^p_{ij} = \frac{1}{c} \, d\varepsilon^p_{kl} \, d\sigma_{kl}.$$

For uniaxial tensile loading using the condition of plastic incompressibility, the above equation implies

$$\frac{d\sigma_{11}}{d\varepsilon^p_{11}} = \frac{3}{2} \, c,$$

where direction 11 corresponds to the direction of the tensile loading. In the classical linear hardening theory of MELAN (1938) and PRAGER (1955, 1956), c is considered as a constant corresponding to bilinear stress-strain curves. It is, of course, of importance to be able to simulate more general stress-strain curves. Even though this can be easily accomplished for increasing proportional loading, most of the proposed nonlinear hardening functions then imply an unrealistic response to reversed loading as, for instance, the proposals of KADASHEVISH and NOVOZHILOV (1958), ZIEGLER (1959), and EISENBERG and PHILLIPS (1968). ISAKSON et al. (1967) and later ARMEN et al. (1971) made the important assumption that the response to reversed loading should be identical to that following during initial loading, but only the increasing proportional loading and the completely reversed loading were treated. RASHID (1974) generalized the above assumption by assuming that the hardening function is initially a function of α_{ij} and that later - for reversed loading - it is a function of α^*_{ij}, where α^*_{ij} is defined by

$$\alpha^*_{ij} = \int_{X_1}^{X} d\alpha_{ij}. \tag{3.1}$$

X corresponds to the actual stress point, while X_1 originally corresponds to the initiation of plastic behaviour and later to the situation where the last reversed loading is initiated. Fig. 2 illustrates the integration path for two cases of uniaxial loading.

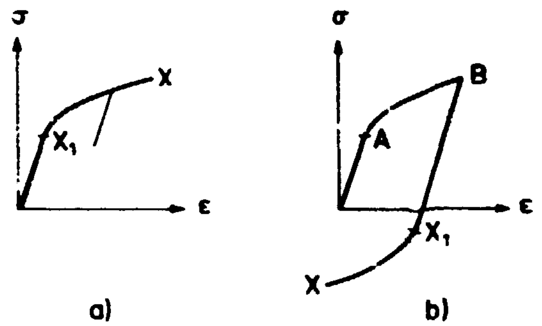

Fig. 2. Integration path for two cases of uniaxial loading.

Fig. 2a) shows that elastic unloading and reloading do not influence the integration path, while Fig. 2b) shows the case of reversed loading. For uniaxial loading, the definition of reversed loading is obvious, but for general multi-axial loading several definitions are possible and proposals have been made by PUGH et al. (1972, and RASHID (1974). Here we adopt the latter, which states that reversed loading is initiated if

$$\alpha^*_{ij} \quad d\sigma_{ij} < 0 \tag{3.2}$$

together with the condition that plastic deformation occurs. RASHID (1974) made a proposal for a nonlinear hardening function involving (3.1) and (3.2), but a discontinuous material behaviour, which seems unrealistic, follows for loadings that gradually deviate more and more from proportional loading and ultimately result in reversed loading. A more detailed discussion of different proposals for nonlinear hardening functions can be found in OTTOSEN (1977).

4. DIRECTION-DEPENDANT NONLINEAR HARDENING

Earlier proposed plasticity models have all assumed that hardening at a specific stress point does not depend on the direction of the new incremental loading $d\sigma_{ij}$. Let us now introduce a model that considers this direction. We assume the following hardening function applicable both to Melan-Prager's and to Ziegler's hardening rule

$$c = c[(\bar{\alpha}^D_{ij}\ \bar{\alpha}^D_{ij})^{\frac{1}{2}}], \tag{4.1}$$

where index D denotes that it is the deviatoric part of the tensor, and where the tensor $\bar{\alpha}_{ij}$ is defined by

$$\bar{\alpha}_{ij} = \alpha^*_{ij} \frac{\alpha^*_{kl}}{(\alpha^*_{st}\ \alpha^*_{st})^{\frac{1}{2}}} \frac{d\sigma_{kl}}{(d\sigma_{mn}\ d\sigma_{mn})^{\frac{1}{2}}} . \tag{4.2}$$

α^*_{kl} is given by (3.1) and the initiation of reversed loading is given by (3.2). (4.2) can be written as $\bar{\alpha}_{ij} = \alpha^*_{ij}\cos\theta$, where θ is the angle between α^*_{kl} and $d\sigma_{kl}$. As a result of (3.2), the angle θ is located in the range $-\pi/2 \leq \theta \leq \pi/2$, i.e. $0\leq\cos\theta \leq 1$ and for increasing proportional loading and for plastic deformation occurring after completely reversed loading we have $\cos\theta = 1$, and thereby $\bar{\alpha}_{ij} = \alpha^*_{ij}$. While α^*_{ij} contains the history of the material, the factor $\cos\theta$ considers the direction of the new incremental loading $d\sigma_{ij}$, whereby a completely smooth behaviour is assured, even when loading deviates more and more from proportional loading and ultimately results in reversed loading.

Considering increasing proportional loading in general, it will be shown that kinematic hardening and isotropic hardening give identical results, provided that the same uniaxial stress-strain curve is used for the calibration of the hardening functions. It is easily shown that (2.15) also applies to isotropic hardening, if s'_{ij} and σ_o are replaced by s_{ij} and σ_e, respectively, where σ_e is the effective stress defined by $\sigma_e = (3s_{ij}s_{ij}/2)^{\frac{1}{2}}$. In addition, the term $3c/2$ should be replaced by $d\sigma_e/d\varepsilon^p_e$, where the differential of the equivalent plastic strain is defined by $d\varepsilon_e^p = (2d\varepsilon_{ij}^p d\varepsilon_{ij}^p/3)^{\frac{1}{2}}$. As we only consider increasing proportional loading, we have $(s_{ij}/\sigma_e)_{iso} = (s'_{ij}/\sigma_o)_{kin}$, i.e. kinematic and isotropic hardening give identical results if the terms $3c/2$ and $d\sigma_e/d\varepsilon_e^p$ are identical for a given stress state. However, this is certainly true, as both terms correspond to the slope at the same point on the uniaxial stress-plastic strain curve that was adopted for calibration, because the following unique relation exists for proportional loading, namely

$$\sigma_e = \sigma_o + (\tfrac{3}{2}\overset{-D}{\alpha}_{ij}\,\overset{-D}{\alpha}_{ij})^{\frac{1}{2}}.$$

5. EXPERIMENTAL VERIFICATION

For uniaxial loading, adoption of (4.1) implies that the stress-strain curve for reversed loading has the same form as the curve for initial plastic loading. This is indicated in Fig. 2b), where curve X_1X has the same form as curve AB. Consequently, the hysteresis curve for zero mean stress or strain is point symmetric around the origin, and this behaviour is in fact the actual experimental behaviour in the steady-state stage (see KREMPL (1971)).

Below, theoretical results are compared with experimental results obtained by means of different combinations of tension and torsion applied to a thin-walled tube. The experimental data are the benchmark tests performed at Oak Ridge National Laboratory and the material is stainless steel type 304 under room temperature conditions. Fig. 3 shows the loading path involving non-proportional loading beyond point A.

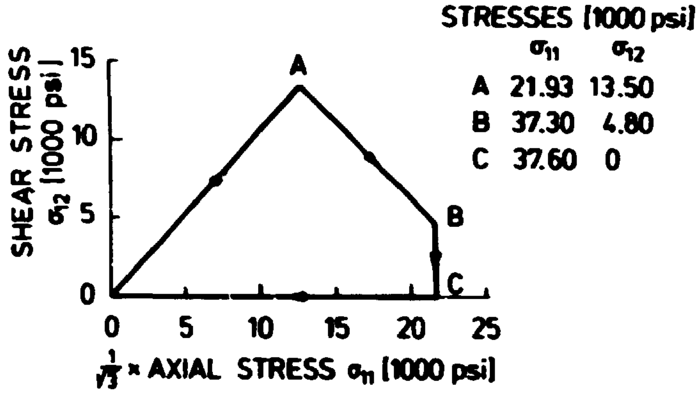

Fig. 3. Loading path for combined tension and torsion.

Three theoretical models were investigated: th- nonlinear kinematic model of (4.1), the classical linear kinematic model and an isotropic model. The approximation to the uniaxial experimental stress-strain curve given by CORUM (1975) is shown in Fig. 4.

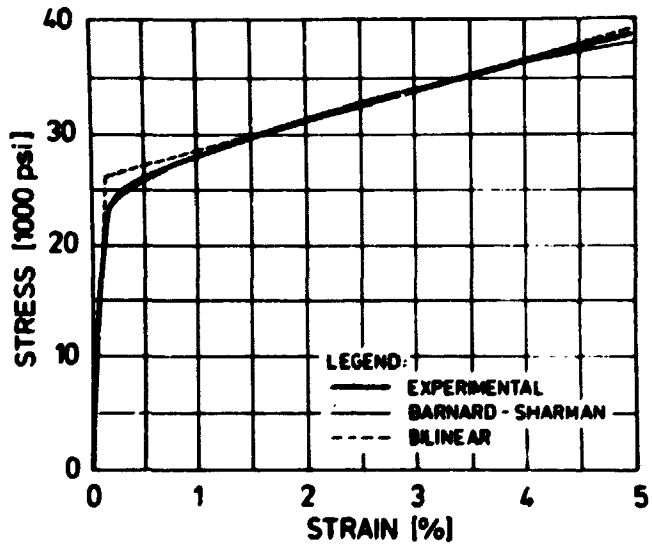

Fig. 4. Uniaxial experimental stress-strain curve, CORUM (1975), and analytical approximations.

The smooth nonlinear approximation to the uniaxial experimental stress-strain curve applied both in the nonlinear kinematic model and in the isotropic model is that proposed bv BARNARD and SHARMAN (1976) stating that

$$\sigma \leq \sigma_o \quad \varepsilon^P = 0; \qquad \sigma \geq \sigma_o \quad \varepsilon^P = A(\sigma - \sigma_o)^B \tag{5.1}$$

where A and B are parameters. Measuring stresses and strains in psi and $\mu\varepsilon$, respectively, the parameter values of (5.1) are: $\sigma_o = 22 \cdot 10^3$, $A = 2.067 \cdot 10^{-9}$ and $B = 1.7477$. The bilinear approximation is obtained by: $\sigma_o = 26.25 \cdot 10^3$ psi and $c = 0.189 \cdot 10^6$ psi. In addition we adopt a Young's modulus of $28.3 \cdot 10^6$ psi and a Poisson's ratio of 0.3.

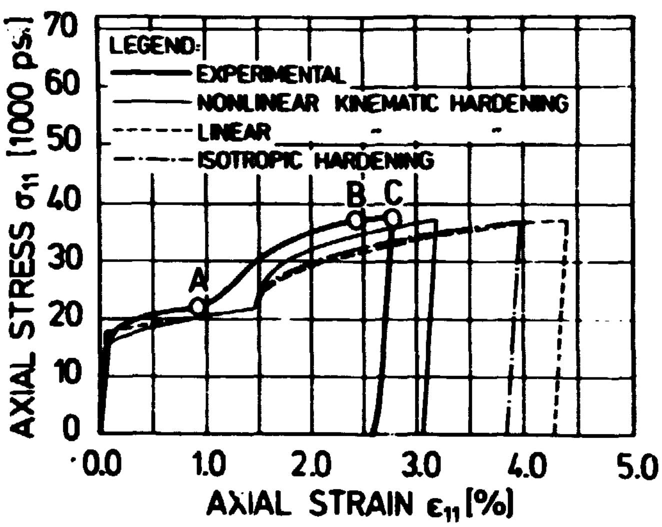

Fig. 5. Experimental axial values, LIU (1975), and corresponding model predictions for combined tension and torsion.

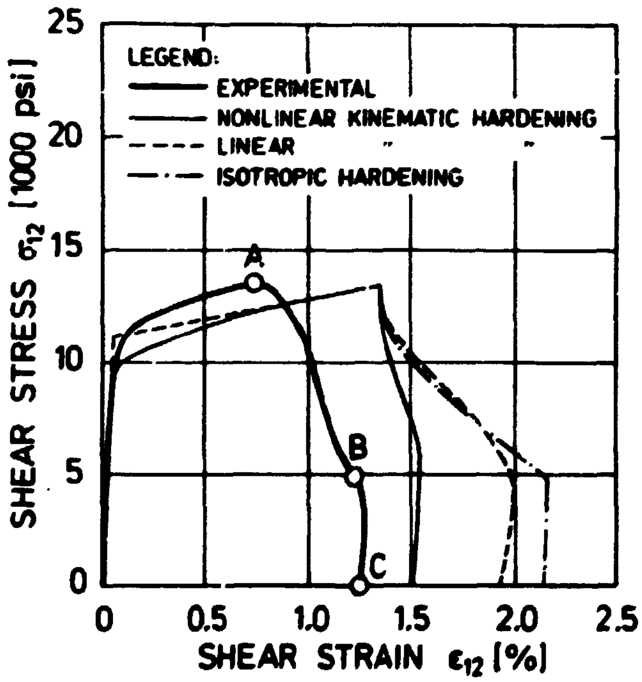

Fig. 6. Experimental shear values, LIU (1975), and corresponding model predictions for combined tension and torsion.

The predictions of the three models and the actual experimental response as reported by LIU (1975) are given in Fig. 5 for the axial values, while the corresponding values for the shear are given in Fig. 6. It is of interest to note that some disagreement exists even for the proportional loading up to point A.

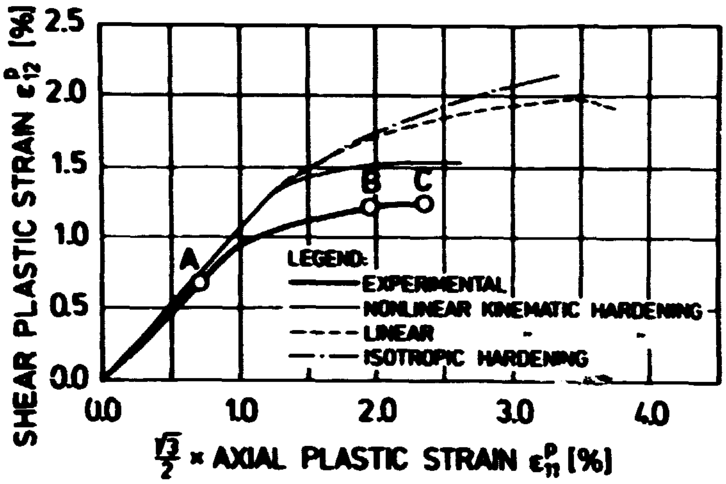

Fig. 7. Experimental strain values, LIU (1975), and corresponding model predictions for combined tension and torsion.

Fig. 7 gives the results of Figs. 5 and 6 in a different form. As before, the nonlinear kinematic model gives the best approximation to the experimental results.

6. CONCLUSIONS AND DISCUSSION

It has been shown that the only cases where Melan-Prager's and Ziegler's hardening rules result in the same material behaviour are those where the yield surface is a sphere in the stress space and where it is a cylinder with a circular cross section. For the latter yield surface, the hardening function must be a type that does not depend on the component of α_{ij} in the direction of the cylinder axis, if this coincidence is to exist. Next, assuming initial isotropy and adopting the von Mises criterion as the initial yield criterion, a nonlinear kinematic hardening function of this type has been proposed for the prediction of metal behaviour. The proposed model has a number of desirable theoretical properties: it predicts identical material behaviour, whether Melan-Prager's or Ziegler's hardening rule is applied; any nonlinear uniaxial stress-strain curve can be used for calibration of the hardening function; general reversed loading is considered and a realistic reversed response is obtained; a smooth change in behaviour is assured when loading deviates more and more from proportional loading; finally, for proportional loading, the response is identical with the response predicted by the corresponding isotropic model. A comparison with experimental data has demonstrated the superiority of the proposed nonlinear kinematic hardening function compared both to the classical linear kinematic model and to the isotropic model. This superiority will be even more pronounced for cyclic loading, but in its present form the proposed model is unable to consider the cyclic hardening or softening observed in several metals. However, even though the proposed model has obvious advantages, the agreement with the experimental data is not overwhelming, and important indications of this discrepancy can be obtained by critically considering some of the assumptions made in the theory. Note that most of the disagreement seems to accumulate during proportional loading. The assumption that initial yield is determined by the von Mises criterion is an excellent approximation as reported by numerous experimenters testing a broad class of metals and as shown by LIU (1975) for the actual material. The normality condition also seems reasonably experimentally verified both for the stainless steel considered, LIU (1975), and for other materials, e.g. PHILLIPS and RICCIUTI (1976). However, when selecting a kinematic hardening model an assumption is introduced that is of much more approximative nature. In particular, even though LIU (1975), for the actual material, and also PHILLIPS and TANG (1972) and PHILLIPS and KASPER (1973), for aluminium, verified a change of the loading surface consisting primarily of a rigid-body translation as assumed in the kinematic hardening theory, they also conclude that the shape of the loading surface changes considerably. This is especially pronounced in the experiments of PHILLIPS and TANG (1972) and PHILLIPS and KASPER (1973), who found that this change of shape is characterized by a lack of cross effect and flattening of part of the loading surface opposite to the loading point. In these experiments it was also found that the rigid-body motion of the loading surface was more in the direction of actual loading than in the direction given by the kinematic hardening theory. By introducing further hardening parameters, some authors, e.g. BALTOV and SAWCZUK (1965) and TANAKA and MIYAGAWA (1975), have constructed kinematic hardening models that also account for some kinds of distortion of the shape of the loading surface, but only part of the experimental evidence mentioned above is reflected, and only linear hardening is considered. The nonlinear model proposed here includes only one hardening function that is completely determined once the uniaxial stress-strain curve is known, and the model seems to offer predictions of sufficient accuracy for most engineering applications, while being at the same time easy to implement in any computer program.

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